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ON THE THERMO-ELASTICITY PROBLEM OF NON-UNIFORM PLATES†

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The general case of the problem of the thermo-elasticity of non-uniform plates is considered. A formal asymptotic expansion is constructed and the limiting problem (when the thickness of the plate approaches zero) is obtained. The limiting problem in the general case turns out to be different from the classical one, in particular, it contains five unknown functions, and the defining equations contain not only the temperature but also its derivatives (although the material of the plate is assumed to obey the Duhamel–Neumann law). These effects do not occur in uniform plates of constant thickness. This is obviously the reason why the effects stated below have not been mentioned previously, as far as we know.

A GENERAL scheme of the asymptotic method for passage from a three-dimensional problem of the theory of elasticity in a thin region (thickness $\varepsilon \ll 1$) to a problem in the theory of plates was previously proposed in [1]. A case which leads to the classical equations of thermo-elastic plates was considered in [2] (it turns out that it corresponds to the case when the coefficients of thermal expansion of the material of the plate are of the order of ε).

1. FORMULATION OF THE PROBLEM

Suppose a three-dimensional linearly elastic body occupying the region Q_ε of characteristic thickness $\varepsilon \ll 1$ is obtained by repetition of an element P_ε (the periodicity cells, PC) in the x_1x_2 plane (Fig. 1). The condition $\varepsilon \ll 1$ is formalized in the form $\varepsilon \to 0$.

The equations of equilibrium of this body have the form [3]

$$\int_{Q_\varepsilon} \sigma_{ij} v_{i,j}\, d\mathbf{x} + \int_{Q_\varepsilon} \bar{f} \bar{v}\, d\mathbf{x} = 0 \tag{1.1}$$

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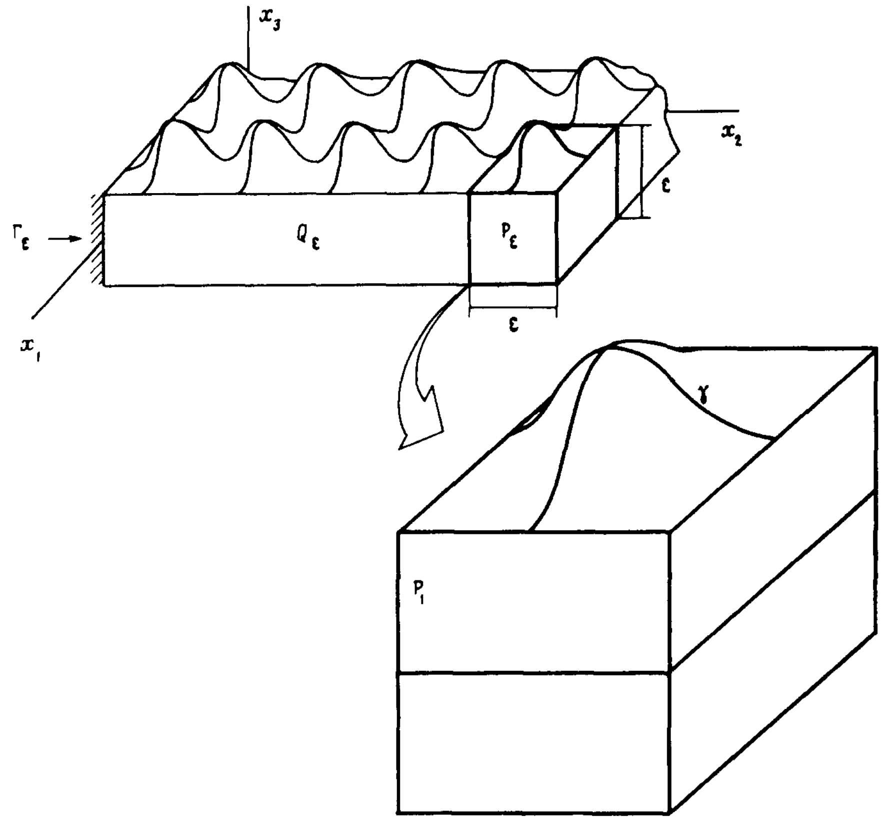

FIG. 1.

$\forall \bar{v} \in V = \{\bar{v} \in H^1(Q_\varepsilon) : \bar{v}(\bar{x}) = 0$ on $\Gamma_\varepsilon\}$, and Γ_ε is the end surface (Fig. 1). Problem (1.1) corresponds to clamping of the body on the end surface and to the free side surface.

We will take the defining equations of the material of the plate in the form (the Duhamel–Neumann law [3])

$$\sigma_{ij} = \varepsilon^{-3}(a_{ijkl}(\bar{x}/\varepsilon)u_{k,l} - \beta_{ij}(\bar{x}/\varepsilon)\theta)$$

$$\beta_{ij}(\bar{x}/\varepsilon) = a_{ijkl}(\bar{x}/\varepsilon)\alpha_{kl}(\bar{x}/\varepsilon) \tag{1.2}$$

where σ_{ij} are the local stresses, $\bar{u}$ are the displacements, θ is the temperature (in the problem considered it is assumed to be a specific steady-state value) and $a_{ijkl}(\bar{x}/\varepsilon)$ and $\alpha_{ij}(\bar{x}/\varepsilon)$ are the tensors of the coefficients of the elastic constants and thermal expansion. The functions $a_{ijkl}(\bar{x}/\varepsilon)$, $\alpha_{ij}(\bar{x}/\varepsilon)$ ($\beta_{ij}(\bar{x}/\varepsilon)$), respectively, are periodic with respect to $\bar{x}$ with a periodicity cell P_ε. Here we take $\beta_{ij}(\bar{x}/\varepsilon)$ in the form

$$\beta_{ij}(\bar{x}/\varepsilon) = \beta_{ij}^{(0)}(\bar{x}/\varepsilon) + \varepsilon\beta_{ij}^{(1)}(\bar{x}/\varepsilon) + \dots \tag{1.3}$$

($\alpha_{ij}(\bar{x}/\varepsilon) = \alpha_{ij}^{(0)}(\bar{x}/\varepsilon) + \varepsilon\alpha_{ij}^{(1)}(\bar{x}\varepsilon) + \dots$, respectively).

Equations (1.2) and (1.3) introduce the parameter ε—the characteristic thickness of the plate—into the defining relations. The following treatments of the presence of ε in them are possible.

The physical asymptotic form. The elastic and thermo-elastic constants of the material are variable, i.e. we consider the spectrum of plates of different thickness and of different materials. For this case a characteristic value of $\beta_{ij}^{(0)} = 0$. This is due to the fact that from physical points of view, as the stiffness of the material increases, its coefficient of thermal expansion should decrease. This case, for $\beta_{ij}^{(1)} \neq 0$, leads to the classical problem of thermo-elasticity [2]. Note that physical representations do not guarantee that $\beta_{ij}^{(k)} \neq 0$. Rather, from the physical point of view we might expect that $\beta_{ij}^{(k)} = 0$ for $k = 0, 1, 2$ and $\beta_{ij}^{(k)} \neq 0$ beginning with $k = 3$.

The geometrical asymptotic form. Suppose the material of the plate is physically the same for all values of its thickness. Then ε in (1.2) and (1.3) is understood as a formal parameter introduced in order to take into account the connection between the stiffness of the plate and its thickness. For this case, $\beta_{ij}^{(0)} \neq 0$ and $\beta_{ij}^{(k)} = 0$ for $k \geq 1$.

2. THE ASYMPTOTIC EXPANSION

To analyse the problem in question we will use the standard asymptotic expansion from [1] for the solution

$$\bar{u} = \bar{u}^{(0)}(\bar{X}) + \varepsilon \bar{u}^{(1)}(\bar{X}, \bar{y}) + \ldots = \sum_{k=0}^{\infty} \varepsilon^k \bar{u}^{(k)}(\bar{X}, \bar{y}) \tag{2.1}$$

$$\langle \bar{u}^{(k)} \rangle = 0, \quad k = 1, 2, \ldots$$

$$\langle \cdot \rangle = \frac{1}{mes\, S_1} \int_{P_1} \cdot\, d\bar{y}$$

is the average of the periodicity cell $P_1 = \varepsilon^{-1} P_\varepsilon = \{\bar{y} = \bar{x}/\varepsilon : \bar{x} \in P_\varepsilon\}$ in the "fast" variables $\bar{y} = \bar{x}/\varepsilon$ and S_1 is the projection of P_1 on the $y_1 y_2$ plane;

for the test function

$$\bar{v} = \bar{v}^{(0)}(\bar{X}) + \varepsilon \bar{v}^{(1)}(\bar{X}, \bar{y}) + \ldots = \sum_{k=0}^{\infty} \varepsilon^k \bar{v}^{(k)}(\bar{X}, \bar{y}) \tag{2.2}$$

for the stresses

$$\sigma_{ij} = \varepsilon^{-3} \sigma_{ij}^{(-3)}(\bar{X}, \bar{y}) + \varepsilon^{-2} \sigma_{ij}^{(-2)}(\bar{X}, \bar{y}) + \ldots = \sum_{m=-3}^{\infty} \varepsilon^m \sigma_{ij}^{(m)}(\bar{X}, \bar{y}) \tag{2.3}$$

Here $\bar{X} = (x_1, x_2)$ is the "slow" variable in the plane of the plate, see [1]. The functions on the right-hand sides of (2.1)–(2.3) are taken to be periodic in y_1 and y_2 with a periodicity cell S_1.

For the functions of the arguments $\bar{X}, \bar{y}$, the differentiation operators $\partial/\partial x_i$ can be represented in the form (see [13]) $\partial/\partial X_\alpha + \varepsilon^{-1}\partial/\partial y_\alpha$ $(\alpha = 1, 2)$, $\varepsilon^{-1}\partial/\partial y_3$.

Here and henceforth the Greek subscripts take the values 1 and 2, and the Latin subscripts take the values 1, 2 and 3 (unless explicitly stated otherwise). We also use the notation $f_{,jy} = \partial f/\partial y_j$, $f_{,\alpha x} = \partial f/\partial x_\alpha$.

Substituting expansions (2.1)–(2.3) into (1.1) and taking into account the rule for the replacement of differentiation operators, we obtain

$$\sum_{k=0}^{\infty} \sum_{m=-3}^{\infty} \int_{Q_1^\varepsilon} \varepsilon \left(\varepsilon^{m+k-1} \sigma_{ij}^{(m)} v_{i,jy}^{(k)} + \varepsilon^{m+k} \sigma_{i\alpha}^{(m)} v_{i,\alpha x}^{(k)}\right) d\bar{v} + \sum_{k=0}^{\infty} \int_{Q_1^\varepsilon} \varepsilon \varepsilon^k \bar{f} \bar{v}^{(k)}\, d\bar{v} = 0 \tag{2.4}$$

Here also we have changed to the variable $\bar{v} = (x_1, x_2, y_3)$ in the integrals. This replacement transfers the region of integration Q_ε of variable thickness into the region $Q_1^\varepsilon = \{(x_1, x_2, y_3 = x_3/\varepsilon) : \bar{x} \in Q_\varepsilon\}$—thicknesses of the order of unity. This is more convenient for investigating the problem. The factor ε in the integrals in (2.4) is related to this replacement.

By substituting the expansions (2.2) and (2.3) into (1.2) we obtain, after equating expressions in the same powers of ε, the following relation:

$$\sigma_{ij}^{(m)} = a_{ijk\alpha} u_{k,\alpha x}^{(m+2)} + a_{ijkl} u_{k,ly}^{(m+3)} - \beta_{ij}^{(m+3)} \theta \tag{2.5}$$

The following relation will be useful later:

$$\int\limits_{Q_1^\varepsilon} f(\bar{X},\bar{y})\,d\bar{v} \to \int\limits_{S} \langle f\rangle(\bar{X})\,dX \text{ при } \varepsilon\to 0 \tag{2.6}$$

where S is the projection of Q_1^ε (and Q_ε) onto the x_1x_2 plane. For the justification of this relationship see [1].

3. THE EQUATIONS OF EQUILIBRIUM OF THE PLATE

These equations are obtained independently of the defining equations. They were obtained in [1], so we will not derive them in detail here. We will merely note that the equations for the forces $N_{i\alpha}^{(m)} = \langle\sigma_{i\alpha}^{(m)}\rangle$ and moments $M_{i\alpha}^{(m)} = \langle\sigma_{i\alpha}^{(m)}\,y_3\rangle$ are obtained from (2.4) by considering test functions of the form $\bar{v} = \bar{v}^{(0)}(\bar{X})$ (the equations for the forces) and $\bar{v} = \varepsilon\bar{v}^{(1)} = \varepsilon y_3\bar{v}_0(\bar{X})$ (the equations for the moments). The equations for the forces have the form

$$N_{i\alpha,\alpha x}^{(m)} = 0,\ m = -3, -2, -1 \tag{3.1}$$

The equations for the moments have the form

$$-M_{i\alpha,\alpha x}^{(m)} + N_{i3}^{(m+1)} = 0,\quad m = -2 \tag{3.2}$$

These equations of equilibrium are identical with the classical ones and are the same irrespective of the defining relations of the materials of the plate [1]. The specific features of the problem considered here arise when analysing the defining relations derived below.

4. DERIVATION OF THE DEFINING RELATIONS OF THE PLATE

4.1. Stretching in the plane of the plate

We will take a test function in Eq. (2.4) in the form $\bar{v} = \varepsilon\bar{v}^1(\bar{y})$. Then $v_{i,\alpha x}^{(1)} = 0$ and in (2.4) the only terms that remain are those corresponding to $k = 1$. Consider the expressions in (2.4) for the same non-negative powers of ε. For $m = -3$, integration of (2.4) by parts gives

$$\sigma_{ij,jy}^{(-3)} = 0 \text{ in } Q_1,\ \sigma_{ij}^{(-3)}n_j = 0 \text{ on } \gamma \tag{4.1}$$

Here $Q_1 = \varepsilon^{-1}Q_\varepsilon$ (not to be confused with Q_1^ε) is obtained by periodic repetition of the periodicity cell P_1 in the y_1y_2 plane, $\bar{n}$ is the external normal to Q_1, and γ is the side (free) surface of Q_1, obtained by periodic repetition of the side surface of the periodicity cell P_1 (in this connection henceforth γ is used to denote both surfaces).

Consider (2.5) once again. For $m = -3$ these relations give

$$\sigma_{ij}^{(-3)} = a_{ijk\alpha}u_{k,\alpha x}^{(0)} + a_{ijkl}u_{k,ly}^{(1)} - \beta_{ij}^{(0)}\theta \tag{4.2}$$

As a result, we arrive at problem (4.1), (4.2) with the following conditions, which arise from the definition of the function $\bar{u}^{(k)}$ in expansion (2.1):

$$\bar{u}^{(1)}(\bar{X},\bar{y}) \text{ are periodic in } y_1, y_2 \text{ with PC } S_1 \text{ and } \langle\bar{u}^{(1)}\rangle=0 \tag{4.3}$$

To solve this problem (which is linear in the variables $\bar{y}$ with $\bar{u}^{(0)}(\bar{X})$, $\theta(\bar{X})$ playing the role of parameters) we introduced [1, 2] the so-called cell problems (CP), which have the following form: the cell problems of the theory of elasticity for plates

$$(a_{ijkl}N_{k,ly}^{p\alpha\nu} - a_{ijp\alpha}y_3^{\nu})_{,jy} = 0 \text{ in } P_1 \tag{4.4}$$
$$(a_{ijkl}N_{k,ly}^{p\alpha\nu} - a_{ijp\alpha}y_3^{\nu})\,n_j = 0 \text{ on } \gamma$$

$\bar{N}^{p\alpha\nu}(\bar{y})$ are periodic in y_1 and y_2 on S_1 and $\langle\bar{N}^{p\alpha\nu}\rangle = 0$, $\nu = 0, 1$ and the cell problems of thermo-elasticity for plates

$$(a_{ijkl}F^{(\nu)}_{k,ly} - \beta^{(\nu)}_{ij})_{,jy} = 0 \text{ in } P_1$$
$$(a_{ijkl}F^{(\nu)}_{k,ly} - \beta^{(\nu)}_{ij})n_j = 0 \text{ on } \gamma \tag{4.5}$$

$\bar{F}^{(\nu)}(\bar{y})$ are period in y_1 and y_2 on S_1 and $\langle\bar{F}^{(\nu)}\rangle = 0$, $\nu = 0, 1$.

Solutions of problems (4.4) and (4.5) exist and are unique for standard conditions on the elastic and thermo-elastic constants (for example [3]).

It was shown in [1] that

$$\bar{N}^{3\beta 0}(\bar{y}) = -y_3\bar{e}_\beta \tag{4.6}$$

where $\{\bar{e}_\beta\}$ are basis vectors.

By comparing (4.4) and (4.5) with (4.1)–(4.3), taking into account the linearity of the last problem, we obtain (taking (4.6) into account)

$$\bar{u}^{(1)} = \bar{N}^{\alpha\beta 0}u^{(0)}_{\alpha,\beta x} - y_3u^{(0)}_{3,\alpha x}\bar{e}_\alpha + \bar{F}^{(0)}\theta + \bar{w}(\bar{X}) \tag{4.7}$$

The occurrence of the function $\bar{w}(\bar{X})$ is due to the fact that problem (4.1)–(4.3) contains only derivatives with respect to the variable $\bar{y}$.

To determine the functions $u_\alpha^{(0)}$ ($\alpha = 1, 2$) we will use Eqs (3.1) for $m = -3$. Substituting (4.7) into (4.2) we obtain

$$\sigma^{(-3)}_{ij} = a_{ijk\alpha}u^{(0)}_{k,\alpha x} + a_{ijkl}N^{\alpha\beta 0}_{k,ly}u^{(0)}_{\alpha,\beta x} - a_{ij\beta 3}u^{(0)}_{3,\beta x} + a_{ijkl}F^{(0)}_{k,ly}\theta - \beta^{(0)}_{ij}\theta =$$
$$= a_{ij\alpha\beta}u^{(0)}_{\alpha,\beta x} + a_{ijkl}N^{\alpha\beta 0}_{k,ly}u^{(0)}_{\alpha,\beta x} + a_{ijkl}F^{(0)}_{k,ly}\theta - \beta^{(0)}_{ij}\theta$$

Averaging the latter equation over the periodicity cell P_1 we obtain for $ij = \gamma\delta$

$$N^{(-3)}_{\gamma\delta} = \langle\sigma^{(-3)}_{\gamma\delta}\rangle = \langle a_{\gamma\delta\alpha\beta} + a_{\gamma\delta kl}N^{\alpha\beta 0}_{k,ly}\rangle u^{(0)}_{\alpha,\beta x} - \langle\beta^{(0)}_{\gamma\delta} - a_{\gamma\delta kl}F^{(0)}_{k,ly}\rangle\theta \tag{4.8}$$

4.2. Bending

Substituting (4.8) into (3.1) for $m = -3$ and $ij = \gamma\delta$ we obtain the equation

$$(\langle a_{\gamma\delta\alpha\beta} + a_{\gamma\delta kl}N^{\alpha\beta 0}_{k,ly}\rangle u^{(0)}_{\alpha,\beta x} - \langle\beta^{(0)}_{\gamma\delta} - a_{\gamma\delta kl}F^{(0)}_{k,ly}\rangle\theta)_{,\delta x} = 0 \tag{4.9}$$

The boundary conditions for $u_\alpha^{(0)}$, $\alpha = 1, 2$ follow from the original boundary conditions and the expansion (2.1) (see [1] for more details) and have the form

$$u_\alpha^{(0)}(\bar{X}) = 0, \quad \alpha = 1, 2 \text{ on } \partial S \tag{4.10}$$

We will denote the solution of problem (4.9), (4.10), which, unlike the elastic case [1], is not necessarily the zeroth solution, by

$$u_\alpha^{(0)} = (R\theta)_\alpha, \quad \alpha = 1, 2 \tag{4.11}$$

where R is the resolving operator of problem (4.9), (4.10). (In fact the operator in (4.11) depends only on $\nabla\theta$ since the coefficients in (4.9) are constants; see [4].)

The fact that the quantity $R\theta$ does not, in general, vanish, gives rise, in the final case, to all the specific features of the problem considered.

Taking (4.11) into account we can rewrite (4.7) in the form

$$\bar{u}^{(1)} = \bar{N}^{\alpha\beta 0}(R\theta)_{\alpha,\beta x} - y_3u^{(0)}_{3,\beta x}\bar{e}_\beta + \bar{F}^{(0)}\theta + \bar{w}(\bar{X}) \tag{4.12}$$

We will continue our consideration of Eq. (2.4) with test function $\bar{v} = \varepsilon\bar{v}^{(1)}(\bar{y})$ ($k = 1$), only now we will consider terms corresponding to $m = -2$. For these we have

$$\sigma^{(-2)}_{ij,jy} = 0 \text{ in } Q_1, \ \sigma^{(-2)}_{ij}n_j = 0 \text{ on } \gamma \tag{4.13}$$

Relations (2.5) with $m = -2$ give

$$\sigma^{(-2)}_{ij} = a_{ijkl}u^{(2)}_{k,ly} + a_{ijk\alpha}u^{(1)}_{k,\alpha x} - \beta^{(1)}_{ij}\theta \tag{4.14}$$

Substituting (4.12) into (4.14) we obtain the equation

$$\sigma_{ij}^{(-2)} = a_{ijkl}u_{k,ly}^{(2)} + a_{ijk\gamma}N_k^{\alpha\beta 0}(R\theta)_{\alpha,\beta x\gamma x} -$$
$$- a_{ij\alpha\beta}y_3 u_{3,\alpha x\beta x}^{(0)} + a_{ijk\alpha}F_k^{(0)}\theta_{,\alpha x} + a_{ijk\alpha}w_{k,\alpha x} - \beta_{ij}^{(1)}\theta \tag{4.15}$$

In addition, according to the definition of the functions $\bar{u}^{(k)}$ from expansion (2.1), we conclude that

$$\bar{u}^{(2)}(\bar{X},\bar{y}) \text{ is periodic in } y_1,\ y_2 \text{ on } S_1 \text{ and } \langle\bar{u}^{(2)}\rangle=0 \tag{4.16}$$

The solution of problem (4.13), (4.15) and (4.16) is again obtained by introducing cell problems. In this case this is the second cell problem of thermo-elasticity

$$(a_{ijkl}T_{k,ly}^{\alpha\beta\gamma} + a_{ijk\gamma}N_k^{\alpha\beta 0})_{,jy} = 0 \text{ in } P_1 \tag{4.17}$$
$$(a_{ijkl}T_{k,ly}^{\alpha\beta\gamma} + a_{ijk\gamma}N_k^{\alpha\beta 0})n_j = 0 \text{ on } \gamma$$

$\bar{T}^{\alpha\beta\gamma}(\bar{y})$ is periodic in y_1 and y_2 on S_1 and $\langle\bar{T}^{\alpha\beta\gamma}\rangle = 0$ and the third problem of thermo-elasticity

$$(a_{ijkl}G_{k,ly}^{(\alpha)} + a_{ijk\alpha}F_k^{(0)})_{,jy} = 0 \text{ in } P_1 \tag{4.18}$$
$$(a_{ijkl}G_{k,ly}^{(\alpha)} + a_{ijk\alpha}F_k^{(0)})n_j = 0 \text{ on } \gamma$$

$\bar{G}^{(\alpha)}(\bar{y})$ is periodic in y_1 and y_2 on S_1 and $\langle\bar{G}^{(\alpha)}\rangle = 0$.

After this the solution of (4.13), (4.15) and (4.16) is obtained in the form

$$\bar{u}^{(2)} = \bar{N}^{\alpha\beta 1}u_{3,\alpha x\beta x}^{(0)} + \bar{T}^{\alpha\beta\gamma}(R\theta)_{\alpha,\beta x\gamma x} + \bar{N}^{\alpha\beta 0}w_{\alpha,\beta x} +$$
$$+ \bar{G}^{(\alpha)}\theta_{,\alpha x} + \bar{F}^{(1)}\theta - y_3 w_{3,\beta x}\bar{e}_\beta \tag{4.19}$$

Substituting (4.19) into (4.15) we obtain

$$\sigma_{ij}^{(-2)} = (a_{ijkl}N_{k,ly}^{\alpha\beta 1} - y_3 a_{ij\alpha\beta})u_{3,\alpha x\beta x}^{(0)} + (a_{ijkl}N_{k,ly}^{\alpha\beta 0} + a_{ij\alpha\beta})w_{\alpha,\beta x} +$$
$$+ (a_{ijkl}T_{k,ly}^{\alpha\beta\gamma} + a_{ijk\gamma}N_k^{\alpha\beta 0})(R\theta)_{\alpha,\beta x\gamma x} + (a_{ijkl}G_{k,ly}^{(\alpha)} + a_{ijk\alpha}F_k^{(0)})\theta_{,\alpha x} +$$
$$+ (a_{ijkl}F_{k,ly}^{(1)} - \beta_{ij}^{(1)})\theta \tag{4.20}$$

The first two terms of relation (4.20) are identical with those obtained in [1] for a purely elastic plate, and the term $\beta_{ij}^{(1)}\theta$ corresponds to [2].

Integrating over the periodicity cell P_1 Eq. (4.20) and the same equation multiplied by y_3, we obtain defining equations connecting the forces and moments $N_{ij}^{(-2)}$ and $M_{ij}^{(-2)}$ with deformation and thermal characteristics.

The defining relations. The elastic and thermo-elastic constants of the plate are the coefficients in the equations connecting the forces and moments with the deformation and temperature characteristics. We will write these.

The elastic constants

$$\mathbf{A}_{ij\alpha\beta}^{\nu+\mu} = \langle(a_{ij\alpha\beta} + a_{ijkl}N_{k,ly}^{\alpha\beta\nu})y_3^{\mu}\rangle \tag{4.21}$$
$$\nu,\ \mu=0,\ 1$$

for $\nu=\mu=0$—the stiffnesses to stretching (in the plane of the plate), for $\nu+\mu=1$—the skew-symmetric part of the stiffnesses, and $\nu=\mu=1$—the stiffnesses to bending.

The thermo-elastic constants

$$\mathbf{B}_{ij}^{(\nu)\mu} = -\langle(\beta_{ij}^{(\nu)} - a_{ijkl}F_{k,ly}^{(\nu)})y_3^{\mu}\rangle \tag{4.22}$$
$$\mathbf{F}_{ij\alpha}^{\mu} = \langle(a_{ijkl}G_{k,ly}^{(\alpha)} + a_{ijk\alpha}F_k^{(0)})y_3^{\mu}\rangle$$
$$\mathbf{T}_{ij\alpha\beta\gamma}^{\mu} = \langle(a_{ijkl}T_{k,ly}^{\alpha\beta\gamma} + a_{ijk\gamma}N_k^{\alpha\beta 0})y_3^{\mu}\rangle$$

with $\mu=0$—the thermo-elastics for deformation in the plane of the plate, and $\mu=1$—for bending.

The defining equations. Using the quantities (4.21) and (4.22) introduced above, we can write [in doing so we take into account expression (4.11)]

$$N_{\gamma\delta}^{(-3)} = \mathbf{A}^0_{\gamma\delta\alpha\beta} u^{(0)}_{\alpha,\beta x} - \mathbf{B}^{(0)0}\theta \tag{4.23}$$
$$N_{\gamma\delta}^{(-2)} = \mathbf{A}^0_{\gamma\delta\alpha\beta} w_{\alpha,\beta x} + \mathbf{A}^1_{\gamma\delta} u^{(0)}_{3,\alpha x\beta x} + \mathbf{T}^{(0)}_{\gamma\delta\alpha\beta\varkappa} u^{(0)}_{\alpha,\beta x\varkappa x} + \mathbf{F}^0_{\gamma\delta\alpha}\theta_{,\alpha x} - \mathbf{B}^{(1)0}_{\gamma\delta}\theta$$
$$M_{\gamma\delta}^{(-2)} = \mathbf{A}^1_{\gamma\delta\alpha\beta} w_{\alpha,\beta x} + \mathbf{A}^2_{\gamma\delta\alpha\beta} u^{(0)}_{3,\alpha x\beta x} + \mathbf{T}^{(1)}_{\gamma\delta\alpha\beta\varkappa} u^{(0)}_{\alpha,\beta x\varkappa x} + \mathbf{F}^1_{\gamma\delta\alpha}\theta_{,\alpha x} - \mathbf{B}^{(1)1}_{\gamma\delta}\theta$$

The equations of equilibrium. We have from (3.1) and (3.2)

$$N^{(-3)}_{\gamma\delta,\delta x} = 0, \quad N^{(-2)}_{\gamma\delta,\delta x} = 0, \quad N^{(-1)}_{3\delta,\delta x} = 0, \quad -M^{(-2)}_{\delta\alpha,\alpha x} + N^{(-1)}_{\delta 3} = 0 \tag{4.24}$$

The boundary conditions. These are obtained in the same way as in [1] by substituting expansion (2.1) into the initial boundary conditions and have the form

$$u^{(0)}_{\sigma}(\bar{X}) = 0, \quad w_\alpha(\bar{X}) = 0, \alpha = 1, 2; \quad u^{(0)}_3(\bar{X}) = \frac{\partial u^{(0)}_3(\bar{X})}{\partial \bar{n}} = 0 \text{ on } \partial S \tag{4.25}$$

Problem (4.23)–(4.25) is the asymptotic version of the problem of the thermo-elasticity of thin plates. It is to a large extent analogous to the classical model but it is not completely identical.

The number of unknowns in the thermo-elasticity problem. In the case of the purely elastic problem, the quantities $u_\alpha^{(0)}$ and w_α ($\alpha = 1, 2$) satisfy the same equations, see [1], as a result of which they may be identified. The solution of the bending problem, correspondingly, can be found in terms of the vector $(w_1, w_2, u_3^{(0)})$, which can be treated as the classical three-dimensional displacement vector [although in the initial meaning its components are the elements of the expansion (2.1) of the displacement vector in problem (1.1)]. In the case of the thermo-elasticity problem, even when there is no bending (when $u_3^{(0)} = 0$) and non-classical effects (see below) $u_\alpha^{(0)}$ and w_α ($\alpha = 1, 2$) cannot be identified, since they satisfy different defining equations [with thermal expansion constants $\mathbf{B}^{(0)0}_{\gamma\delta} \neq \mathbf{B}^{(1)0}_{\gamma\delta}$ for $N^{(-3)}_{\gamma\delta}$ and $N^{(-2)}_{\gamma\delta}$, see (4.23)]. Hence, the five-dimensional vector $(u_1^{(0)}, u_2^{(0)}, w_1, w_2, u_3^{(0)})$ acts as the solution of the thermo-elasticity problem for plates. In this case the problem in terms of $(u_1^{(0)}, u_2^{(0)})$—the displacements in the plane of the plate, is not related to the problem in terms of $(w_1, w_2, u_3^{(0)})$, which are displacements of a bending nature and can be solved independently of the latter (but not, in general, vice versa).

5. THE GEOMETRICAL AND PHYSICAL ASYMPTOTIC FORMS

1. In the case of the physical asymptotic form, when $\beta^{(0)}_{ij} = 0$, but virtue of (4.5) and (4.18) we have that $\bar{F}^{(0)} = \bar{G}^{(\alpha)} = 0$, and by virtue of $R\theta = 0$ [see (4.11)] the term with $\bar{T}^{\alpha\beta\gamma}$ does not occur in the defining equations. In this case $u_1^{(0)} = u_2^{(0)} = 0$ and need not be considered, while the defining equations for the problem with respect to $(w_1, w_2, u_3^{(0)})$ are obtained if in (4.23) we formally put $\mathbf{B}^{(\nu)\mu}_{\gamma\delta} = \mathbf{T}^{(\nu)\mu}_{\gamma\delta} = \mathbf{F}^{\mu}_{\gamma\delta\alpha} = 0$. In this case $\mathbf{B}^{(0)0}_{\gamma\delta} = 0$ indicates that the plate on heating does not expand in its plane. This completely agrees with the fact that the coefficient of thermal expansion of the material of the plate $\beta_{ij} = \varepsilon\, \beta^{(1)}_{ij} + \ldots \to 0$ when $\varepsilon \to 0$ [see (1.3)].

2. The geometrical asymptotic form corresponds to the case $\beta^{(0)}_{ij} \neq 0$, which, generally speaking, leads to the presence of all the coefficients in (4.23). This asymptotic form corresponds, in particular, to "normal" thermal expansion of the plate in its plane (according to (1.3), $\beta_{ij} = \beta^{(0)}_{ij}$ in this case).

A uniform plate of constant thickness, the geometrical asymptotic form. Suppose the plate is made of a uniform isotropic material and has constant thickness:

$$a_{ijkl} = \text{const}, \quad \beta_{ij} = \text{const}, \quad P_t = [-1, 1]^2 \times [-{}^1/_2, {}^1/_2]$$

In this case the solutions of problems (4.4) and (4.5) are as follows:

$$N_1^{\alpha\beta 0} = N_2^{\alpha\beta 0} = 0, \quad N_3^{\alpha\beta 0} = -\frac{a_{33\alpha\beta}}{a_{3333}} y_3$$
$$F_1^{(0)} = F_2^{(0)} = 0, \quad F_3^{(0)} = \frac{\beta_{33}}{a_{3333}} y_3$$

Substituting these expressions into (4.17) and (4.18), respectively, and solving the resulting problems, we obtain

$$T_\varkappa^{\alpha\beta\gamma} = 0 \text{ for } \varkappa \neq \gamma, \quad T_\gamma^{\alpha\beta\gamma} = -\frac{a_{33\alpha\beta}}{2a_{3333}}\left(y_3{}^2 - \frac{1}{12}\right)$$
$$G_\varkappa^{(\alpha)} = 0 \text{ for } \varkappa \neq \alpha, \quad G_\alpha^{(\alpha)} = -\frac{\beta_{33}}{2a_{3333}}\left(y_3{}^2 - \frac{1}{2}\right)$$

Substituting these expressions into (4.22) we obtain

$$\mathbf{F}_{ij\alpha}^{\mu} = \left\langle \left(-a_{ij\alpha 3}\frac{\beta_{33}}{a_{3333}} y_3 + a_{ij3\alpha}\frac{\beta_{33}}{a_{3333}} y_3 \right) y_3{}^{\mu} \right\rangle = 0$$
$$\mathbf{T}_{ij\alpha\beta\gamma}^{\mu} = \left\langle \left(-a_{ij\gamma 3}\frac{a_{33\alpha\beta}}{a_{3333}} y_3 + a_{ij3\gamma}\frac{a_{33\alpha\beta}}{a_{3333}} y_3 \right) y_3{}^{\mu} \right\rangle = 0$$

Here we have used the well-known symmetry of the elastic-constant tensor [3]. By making the coefficients $\mathbf{T}_{ij\alpha\beta\gamma}^{\mu}$ zero we "decouple" the problems with respect to $(u_1{}^{(0)}, u_2{}^{(0)})$ and $(w_1, w_2, u_3{}^{(0)})$.

Plates of non-constant thickness made of uniform material. Suppose, as above, that $a_{ijkl} = \text{const}$, $\beta_{ij} = \text{const}$, but P_1 is an arbitrary region (capable of acting as a periodicity cell). In this case, generally speaking, $\mathbf{F}_{ij\alpha}^{\mu}$, $\mathbf{T}^{\mu}{}_{ij\alpha\beta\gamma} \neq 0$.

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